A Riemann-Roch for Number Fields

Brian Ton

UC San Diego

Overview

Motivations

- Algebraic number theory for number fields (finite extensions of $K/\mathbb{Q}$) is typically built up in an ideal-theoretic point of view
  - Dedekind domains, fractional ideals, ideal factorization, etc.
- One other approach is via a valuation-theoretic point of view, using the data from finite places (corresponding to prime ideals of $\mathcal{O}_K$) and infinite places (arising from embeddings of K into $\mathbb{R}$ and $\mathbb{C}$.
  - We will go so far as to look at an "analogue" of the Riemann Roch Theorem for number fields

Basic Definitions

Definition (Prime)

A prime (or place) $\mathfrak{p}$ of an algebraic number field K is a class of equivalent valuations on K. The nonarchimedean equivalence classes of valuations are called finite primes and the archimedean ones infinite primes.

Note: we will denote infinite primes by the notation $\mathfrak{p} \mid \infty$ and the finite ones by $\mathfrak{p} \nmid \infty$.
To each prime $\mathfrak{p}$, we associate a homomorphism $v_{\mathfrak{p}} : K^{\times} \to \mathbb{R}$. If $\mathfrak{p}$ is finite, then take $v_{\mathfrak{p}}$, to be the $\mathfrak{p}$-adic valuation, normalized by $v_{\mathfrak{p}}(K^{\times}) = \mathbb{Z}$. If $\mathfrak{p} \mid \infty$, then $v_{\mathfrak{p}}(a) = -\log|\tau a|$, where $\tau : K \to \mathbb{C}$ is an embedding which defines $\mathfrak{p}$.

Definition (Replete Ideal)

A replete ideal is an element of the group

$$J(\overline{\mathcal{O}_K}) := J(\mathcal{O}_K) \times \prod_{\mathfrak{p}|\infty} \mathbb{R}_+^\times$$

where $\mathbb{R}_+^\times$ denotes the multiplicative group of positive real numbers and $J(\mathcal{O})$ denotes the group of fractional ideals.

An arbitrary element of $J(\overline{\mathcal{O}_K})$ looks like

$$\mathfrak{a} = \prod_{p\nmid\infty} \mathfrak{p}^{v_\mathfrak{p}} \times \prod_{p|\infty} \mathfrak{p}^{v_\mathfrak{p}} =: \prod_{\mathfrak{p}} \mathfrak{p}^{v_\mathfrak{p}}$$

(a tuple) where $v_\mathfrak{p} \in \mathbb{Z}$ for $p \nmid \infty$ and $v_\mathfrak{p} \in \mathbb{R}$ for $\mathfrak{p} \mid \infty$. Here, to unify notation, $\mathfrak{p}^{v_\mathfrak{p}} := e^{v_\mathfrak{p}}$ when $\mathfrak{p} \mid \infty$. Let $\mathfrak{a}_f := \prod_{\mathfrak{p}\nmid\infty} \mathfrak{p}^{v_\mathfrak{p}}$ denote the "finite" part and $\mathfrak{a}_\infty = \prod_{\mathfrak{p}|\infty} \mathfrak{p}^{v_\mathfrak{p}}$ denote the "infinite" part, so that $\mathfrak{a} = \mathfrak{a}_f \times \mathfrak{a}_\infty$.

Replete Picard Group

Continuing from the last slide, to each $a \in K^\times$, associate the replete principal ideal

$$[a] = \prod_{\mathfrak{p}} \mathfrak{p}^{v_\mathfrak{p}(a)} = (a) \times \prod_{\mathfrak{p} | \infty} \mathfrak{p}^{v_\mathfrak{p}(a)}$$

These principal replete ideals form a subgroup $P(\overline{\mathcal{O}_K})$ of $J(\overline{\mathcal{O}_K})$. We can now define the

Definition (Replete Picard Group)

The factor group

$$\mathrm{Pic}(\overline{\mathcal{O}_K}) = J(\overline{\mathcal{O}_K})/P(\overline{\mathcal{O}_K})$$

is called the replete ideal class group, or the replete Picard group.

Notice the analogy with the "regular" ideal class group!

Replete Divisors

Definition (Replete Divisor)

A replete divisor (or Arakelov divisor) of K is a formal sum

$$D = \sum_{\mathfrak{p}} v_{\mathfrak{p}} \mathfrak{p}$$

with $v_{\mathfrak{p}} \in \mathbb{Z}$ for $\mathfrak{p} \nmid \infty$, $v_{\mathfrak{p}} \in \mathbb{R}$ for $\mathfrak{p} \mid \infty$ and $v_{\mathfrak{p}} = 0$ for almost all $\mathfrak{p}$.

Let $\mathrm{Div}(\overline{\mathcal{O}_K})$ denote the group of these replete divisors. Define $\mathrm{div} : K^\times \to \mathrm{Div}(\overline{\mathcal{O}_K})$ to be $a \mapsto \sum_{\mathfrak{p}} v_{\mathfrak{p}}(a)\mathfrak{p}$. The image is the group of replete principal divisors, denoted $\mathcal{P}(\overline{\mathcal{O}})$. Now, define the

Definition (Arakelov Class Group)

$$CH^1(\overline{\mathcal{O}_K}) = \mathrm{Div}(\overline{\mathcal{O}_K})/\mathcal{P}(\overline{\mathcal{O}_K})$$

We can now show that

Proposition (Ideals and Divisors)

$$\mathrm{div} : J(\overline{\mathcal{O}_K}) \to \mathrm{Div}(\overline{\mathcal{O}_K}), \quad \prod_{\mathfrak{p}} \mathfrak{p}^{v_\mathfrak{p}} \mapsto \sum_{\mathfrak{p}} -v_\mathfrak{p}\mathfrak{p}$$

and

$$\mathrm{Div}(\overline{\mathcal{O}_K}) \to J(\overline{\mathcal{O}_K}), \quad \sum_{\mathfrak{p}} v_\mathfrak{p} \mapsto \prod_{\mathfrak{p}} \mathfrak{p}^{-v_\mathfrak{p}}$$

are mutually inverse isomorphisms between $J(\overline{\mathcal{O}_K})$ and $\mathrm{Div}(\overline{\mathcal{O}_K})$, which induce a (topological) isomorphism $\mathrm{div} : \mathrm{Pic}(\overline{\mathcal{O}_K}) \to CH^1(\overline{\mathcal{O}_K})$

What's the point? Much of the theory we developed via ideals can be viewed through the language/theory of divisors (e.g. Dirichlet's Unit Theorem / Finiteness of the Class Group)!

Towards "Riemann-Roch"

To build towards our analogue of the Riemann-Roch theorem, we note that we have a similar setup: recall that a divisor on a Riemann surface X is a finite formal sum $D = \sum_{P \in X} a_P P$, where the a_P encodes order of a pole / zero at P for the principal divisors $\mathrm{div}(f) = \sum_{P \in X} a_P P$. Then,

$$\begin{aligned} H^0(D) &:= H^0(X, \mathcal{O}(D)) \\ &:= \{\text{rational functions } f \text{ on } X \ : \mathrm{ord}_P(f) \geq -a_P\} \end{aligned}$$

and the Riemann-Roch theorem tells us that

$$\dim H^0(D) - \dim H^1(D) = \deg(D) + 1 - g$$

where g is the genus of the curve and the left-hand side $\dim H^0(D) - \dim H^1(D) =: \chi(D)$ is the Euler characteristic of X.

Our Analogues

We define our analogues in a pseudo-ad-hoc way. For each replete ideal $\mathfrak{a} = \prod_{\mathfrak{p}} \mathfrak{p}^{v_{\mathfrak{p}}}$ of K, define

$$\chi(\mathfrak{a}) = -\log \mathrm{vol}(\mathfrak{a})$$

where $\mathrm{vol}(\mathfrak{a}) = \mathcal{N}(\mathfrak{a}_\infty)\mathrm{vol}(\mathfrak{a}_f)$ and

$$H^0(\mathfrak{a}) = \{f \in K^\times : v_{\mathfrak{p}}(f) \geq v_{\mathfrak{p}}, \forall \mathfrak{p}\}$$

(which is a finite set, using some Minkowski Theory) and

$$\dim H^0(\mathfrak{a}) = \log \frac{\# H^0(\mathfrak{a})}{2^r (2\pi)^s}$$

We define the genus of a number field K is defined to be the real number

$$g = g_K = \dim H^0(\mathcal{O}_K) - \chi(\mathcal{O}_K) = \log \frac{\#\mu(K)\sqrt{|d_K|}}{2^r (2\pi)^s}$$

Riemann Roch Theorem

With these definitions, we arrive at the following analogue of the Riemann-Roch theorem:

Theorem (Arithmetic Riemann-Roch)

For every replete ideal $\mathfrak{a}$ of K, one has

$$\chi(\mathfrak{a}) = \deg(\mathfrak{a}) + \dim H^0(\mathcal{O}_K) - g$$

With some extra work (using Minkowski Theory and a theorem due to Lang on the asymptotics of $\# H^0(\mathfrak{a}^{-1})$), one can show the following analogue of the strong Riemann-Roch formula (defining $\dim H^0(D) = \log \frac{\# H^0(D)}{2^r (2\pi)^s}$ and $\chi(D) = \chi(\mathcal{O}(D))$):

$$\dim H^0(D) = \deg(D) + \dim H^0(\mathcal{O}_K) - g + i(D)$$

where $i(D) = O(e^{-\frac{1}{n}\deg(D)})$. In particular, $i(D) \to 0$ as $\deg(D) \to \infty$, so that $i(D)$ is an analogue of $\dim H^1(D)$.

Riemann Hurwitz

We define

Definition (Dedekind's Complementary Module)

Let L/K be a finite extension. Then,

$$\mathscr{C}_{L|K} = \{x \in L : \mathrm{Tr}(x\mathcal{O}_L) \subseteq \mathcal{O}_K\} \simeq \mathrm{Hom}_{\mathcal{O}_K}(\mathcal{O}_L, \mathcal{O}_K)$$

Under this definition, we can obtain the following analogue for the Riemann-Hurwitz formula.

Proposition

For $L|K$ a finite extension, and g_L and g_K are the genus of L and K, respectively, then

$$g_L - \dim H^0(\mathcal{O}_L) = [L:K](g_K - \dim H^0(\mathcal{O}_K)) + \frac{1}{2}\deg \mathscr{C}_{L|K}$$

Grothendieck's Theory

In the previous section, we saw a (very) ad-hoc way to arrive at our Riemann–Roch theorem, a special case of a far more general Grothendieck–Riemann–Roch theorem. We will look at this theory for the case of $\mathcal{O}_K$-modules.

Metrized $\mathcal{O}_K$ Modules

Let $K/\mathbb{Q}$ be a finite extension, and let

$$K_{\mathbb{C}} = K \otimes_{\mathbb{Q}} \mathbb{C}$$

be the complexification of K. Let F be the the generator of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. On an $\mathcal{O}_K$ module M and its complexification $M_{\mathbb{C}} = M \otimes_{\mathbb{Z}} \mathbb{C}$, we define the action of F on $M_{\mathbb{C}}$ by $F(a \otimes x) = a \otimes \overline{x}$. This leads us to the definition

Definition (Metrized $\mathcal{O}_K$ Modules)

A metrized $\mathcal{O}_K$-module is a finitely generated $\mathcal{O}_K$-module M with an F-invariant hermitian metric on $M_{\mathbb{C}}$.

where we note that an F-invariant hermitian metric on $M_{\mathbb{C}}$ is a $K_{\mathbb{C}}$-linear form $\langle , \rangle_M : M_{\mathbb{C}} \times M_{\mathbb{C}} \to K_{\mathbb{C}}$ satisfying $\overline{\langle x, y \rangle}_M = \langle y, x \rangle_M$ and $F\langle x, y \rangle_M = \langle Fx, Fy \rangle_M$

Denote by $\{M\}$ the isometry class of a metrized $\mathcal{O}_K$-module M. Now, we define

$$F_0(\overline{\mathcal{O}_K}) = \bigoplus_{\{M\},\, M \text{ proj.}} \mathbb{Z}\{M\}, \quad F^0(\overline{\mathcal{O}_K}) = \bigoplus_{\{M\},\, M \text{ f.g.}} \mathbb{Z}\{M\}$$

Definition (Grothendieck Groups)

The quotient groups

$$K_0(\overline{\mathcal{O}_K}) = F_0(\overline{\mathcal{O}_K})/R_0(\overline{\mathcal{O}_K}), \quad K^0(\overline{\mathcal{O}_K}) = F^0(\overline{\mathcal{O}_K})/R^0(\overline{\mathcal{O}_K})$$

are called the replete Grothendieck groups of $\mathcal{O}_K$, where $R_0(\overline{\mathcal{O}_K})$ and $R^0(\overline{\mathcal{O}_K})$ are the subgroups of $F_0(\overline{\mathcal{O}_K})$ and $F^0(\overline{\mathcal{O}_K})$ generated by elements $\{M'\} - \{M\} + \{M''\}$ which arise from a short exact sequence $0 \to M' \to M \to M'' \to 0$ of projective / f.g. metrized $\mathcal{O}_K$-modules.

Notice that under this definition, $\{M \oplus N\} = \{M\} + \{N\}$ in $K_0(\overline{\mathcal{O}_K})$ and $K^0(\overline{\mathcal{O}_K})$!

Grothendieck Groups

There are a few nice properties of the Grothendieck Groups. Firstly, the tensor product induces a ring structure on $K_0(\overline{\mathcal{O}_K})$: $\{M\}\{M'\} = \{M \otimes_{\mathcal{O}_K} M'\}$, and $K^0(\mathcal{O}_K)$ becomes a $K_0(\mathcal{O}_K)$ module. There is an important theorem relating them:

Proposition (The Poincaré Isomorphism)

The map $K_0(\overline{\mathcal{O}_K}) \to K^0(\overline{\mathcal{O}_K})$ given by $\{M\} \mapsto \{M\}$ defines a homomorphism (called the Poincaré homomorphism). In particular, this is a $K_0(\overline{\mathcal{O}_K})$-module isomorphism.

The Grothendieck ring $K_0(\overline{\mathcal{O}_K})$ is equipped with a canonical surjective homomorphism $\mathrm{rk} : K_0(\overline{\mathcal{O}_K}) \to \mathbb{Z}$ given by $\{M\} \mapsto \dim_K(M \otimes_{\mathcal{O}_K} K)$ with $\ker(\mathrm{rk}) = I$. One can show $I^2 = 0$.

Chern Character

Define the graded Grothendieck group

$$\operatorname{gr} K_0(\overline{\mathcal{O}_K}) = \mathbb{Z} \oplus I$$

This becomes a ring after defining $xy = 0$ for $x, y \in I$. The additive homomorphism

$$c_1 : K_0(\overline{\mathcal{O}_K}) \to I, \quad c_1(\{M\}) = \{M\} - \operatorname{rk}(\{M\})$$

is called the first Chern Class. The mapping

$$\operatorname{ch} : K_0(\overline{\mathcal{O}_K}) \to \operatorname{gr} K_0, \quad \operatorname{ch}(\{M\}) = \operatorname{rk}(\{M\}) + c_1(\{M\})$$

is called the Chern character. One can show that $\operatorname{ch} : K_0(\overline{\mathcal{O}_K}) \to \operatorname{gr} K_0(\overline{\mathcal{O}_K})$ is an isomorphism of rings. Under the isomorphism $\det : I \xrightarrow{\sim} \operatorname{Pic}(\overline{\mathcal{O}_K})$, we obtain that $\operatorname{ch} : K_0(\overline{\mathcal{O}_K}) \to \mathbb{Z} \oplus \operatorname{Pic}(\overline{\mathcal{O}_K}) =: \operatorname{CH}(\overline{\mathcal{O}_K})$ is also an isomorphism.

Towards Grothendieck-Riemann-Roch

Now that we know about metrized $\mathcal{O}_K$ modules, we want to figure out the relationship between metrized $\mathcal{O}_L$ and $\mathcal{O}_K$ modules when L/K is a finite extension. Indeed, the inclusion $\mathcal{O}_K \to \mathcal{O}_L$ and $\mathrm{Hom}(L, \mathbb{C}) \to \mathrm{Hom}(K, \mathbb{C}), \sigma \mapsto \sigma|_K$ induce two canonical homomorphisms

$$i^* : K_0(\overline{\mathcal{O}_K}) \to K_0(\overline{\mathcal{O}_L}), \quad i_* : K_0(\overline{\mathcal{O}_L}) \to K_0(\overline{\mathcal{O}_K})$$

where $i^*(\{M\}) = [M \otimes_{\mathcal{O}_K} \mathcal{O}_L]$ and $i_*(\{M\}) = \{M\}$. The Grothendieck-Riemann-Roch theorem's goal is to compute the Chern character $\mathrm{ch}(i_* M)$ for a projective metrized $\mathcal{O}_L$-module in terms of $\mathrm{ch}(M)$.

Math is Ugly

From the definition of i_*, we get the following commuting diagram

$$\begin{array}{ccc} K_0(\overline{\mathcal{O}_L}) & \xrightarrow{\text{rk}} & \mathbb{Z} \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle [L:K]} \\ K_0(\overline{\mathcal{O}_K}) & \xrightarrow{\text{rk}} & \mathbb{Z} \end{array}$$

Therefore, i_* induces a homomorphism $i_* : I(\overline{\mathcal{O}_L}) \to I(\overline{\mathcal{O}_K})$. which further induces $i_* : \operatorname{gr} K_0(\overline{\mathcal{O}_L}) \to \operatorname{gr} K_0(\overline{\mathcal{O}_K})$ called the Gysin map. We now consider the following diagram

$$\begin{array}{ccc} K_0(\overline{\mathcal{O}_L}) & \xrightarrow{\text{ch}} & \operatorname{gr} K_0(\overline{\mathcal{O}_L}) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\ K_0(\overline{\mathcal{O}_K}) & \xrightarrow{\text{ch}} & \operatorname{gr} K_0(\overline{\mathcal{O}_K}) \end{array}$$

But this doesn't commute!

Grothendieck-Riemann-Roch

Fortunately, the situation on the previous slide can be corrected, which will be provided via the module of Kähler differentials $\Omega^1_{\mathcal{O}_L|\mathcal{O}_K} = J/J^2$, where J is the kernel of $\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L \to \mathcal{O}_L$. We now define the Todd class $\mathrm{Td}(\mathcal{O}_L|\mathcal{O}_K) = 1 - \frac{1}{2}c_1([\Omega^1_{\mathcal{O}_L|\mathcal{O}_K}])$ (note that $\mathrm{Td}(\mathcal{O}_L|\mathcal{O}_K) \notin \mathrm{gr} K_0(\overline{\mathcal{O}_L})$ itself). We now have the theorem:

Theorem (Grothendieck-Riemann-Roch)

The diagram

$$\begin{array}{ccc} K_0(\overline{\mathcal{O}_L}) & \xrightarrow{\mathrm{Td}(\mathcal{O}_L|\mathcal{O}_K)\mathrm{ch}} & \mathrm{gr}\, K_0(\overline{\mathcal{O}_L}) \\ \downarrow i_* & & \downarrow i_* \\ K_0(\overline{\mathcal{O}_K}) & \xrightarrow{\mathrm{ch}} & \mathrm{gr}\, K_0(\overline{\mathcal{O}_K}) \end{array}$$

commutes!

Conclusion

What have we accomplished? We have integrated the basic theorems of algebraic number theory as a special case of a more general viewpoint in geometry. In particular, we were looking at the scheme $X = \mathrm{Spec}(\mathcal{O}_K)$, $\mathcal{O}_K$-modules as metrized vector bundles, $\Omega^1_{\mathcal{O}_L|\mathcal{O}_K}$ as the cotangent element, the morphism of schemes $Y = \mathrm{Spec}(\mathcal{O}_L) \to X$, etc.

References I

[NS13] J. Neukirch and N. Schappacher. *Algebraic Number Theory*. Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2013. ISBN: 9783662039830. URL: https://books.google.com/books?id=hS3qCAAAQBAJ.